A non-Archimedean approach to K-stability and the existence of Kähler-Einstein metrics

Mattias Jonsson

University of Michigan

Stability, Boundedness and Fano Varieties, BICMR, August 2017

Plan

1. KE metrics on Fano manifolds and coercivity of functionals.

2. K-stability from several points of view.

3. A variational proof of the YTD conjecture.

4. Extensions and speculations.

Based on joint work with R. Berman, H. Blum, S. Boucksom and T. Hisamoto (in various configurations).

Heavily uses work by many, many other people! References far from complete: apologies!

Part 1: Kähler-Einstein metrics on Fano manifolds

In this part:

- General remarks on Kähler-Einstein metrics.
- The “calculus” of metrics on line bundles.
- Functionals on the space of metrics.
- Kähler-Einstein metrics and coercivity of Ding and Mabuchi.

Kähler-Einstein metrics

- X = smooth complex projective variety of dim n.
- K_X = canonical bundle (or divisor class).
- ω = Kähler form on X. Also think of as Hermitean metric.
- Say ω is a *Kähler-Einstein* (KE) metric if

  $$\mathrm{Ric}\,\omega = \lambda\omega$$

  for some $\lambda \in \mathbf{R}$.
- The cases $\lambda < 0$ (X can. polarized) and $\lambda = 0$ (X Calabi-Yau) are understood due to work by Calabi, Aubin and Yau. Namely, there exists a unique Kähler-Einstein metric.
- Remains to consider the case when $\lambda < 0$ and X is *Fano*, i.e. $-K_X$ ample. After scaling, $\lambda = -1$ so we look at

  $$\mathrm{Ric}\,\omega = \omega.$$

- This equation may or may not have a solution!

KE metrics in Fano case

- Assume X Fano and look at the equation

$$\operatorname{Ric}\omega = \omega, \tag{KE1}$$

where $\omega \in c_1(X) = c_1(-K_X)$ is a Kähler form on X.

- Fix a reference Kähler form $\omega_0 \in c_1(X)$ and write

$$\omega = \omega_0 + dd^c\varphi,$$

for $\varphi \in C^\infty(X)$, where $dd^c = \frac{i}{\pi}\partial\overline{\partial}$. Then (KE1) becomes

$$(\omega_0 + dd^c\varphi)^n = ce^{-2\varphi}\mu, \tag{KE2}$$

where $c = c(\varphi) > 0$ is a normalizing constant and $\mu = \mu(\omega_0)$ is a positive volume form on X.

- Equation (KE2) is a PDE of *Monge-Ampère* type.
- Both existence and uniqueness are nontrivial.

Bando-Mabuchi and YTD

- *Uniqueness* governed by the *Bando-Mabuchi theorem*.
- **Thm** [Bando-Mabuchi; Berndtsson] If X is Fano and ω, ω' are KE metrics, then there exists $g \in \mathrm{Aut}(X)$ such that $\omega' = g^*\omega$.
- *Existence* is more subtle. Starting with Matsushima '57, people found various *obstructions*.
- Example: $\mathbf{P}^2$ blown up in one pt is Fano but has no KE metric.
- *YTD conjecture*: a KE metric exists iff X is *K-(poly)stable*.
- Discuss K-stability later. In principle algebraic condition on X.
- **Thm**[Chen-Donaldson-Sun, Tian] The YTD conjecture is true.

Methods for constructing KE metrics

- Several approaches to solving the equation

$$(\omega_0 + dd^c\varphi)^n = ce^{-2\varphi}\mu. \tag{KE2}$$

  - Cont. method (Chen-Donaldson-Sun, Tian, Székelyhidi).
  - Kähler-Ricci flow (Chen-Sun-Wang).
  - Variational method (Berman-Boucksom-J).
- Will only discuss the variational method in these lectures.
- All methods have versions (easier, but still nontrivial) in the canonically polarized and Calabi-Yau case. In these cases, solutions always exist.
- In the Fano case, solutions do not always exist. The methods must use the K-polystability assumption (explained later).

Variational approach: basic idea

- Again look at the equation

$$(\omega_0 + dd^c\varphi)^n = ce^{-2\varphi}\mu. \tag{KE2}$$

- Consider the space $\mathcal{H} := \{\varphi \in C^\infty(X) \mid \omega_0 + dd^c\varphi > 0\}$.
- Define a functional $F\colon \mathcal{H} \to \mathbf{R}$ whose critical points, $F'(\varphi) = 0$, are solutions to (KE2).
- Find these critical points as *minima* of F on $\mathcal{H}$.
- Not obvious that a minimizer exists: ignore this issue for now.
- Will use two different functionals: Mabuchi and Ding...
- ...as well as some other functionals.
- Useful to identify elements of $\mathcal{H}$ as *metrics* on $-K_X$.

Metrics on line bundles

- Equip $\mathbf{C}$ with the usual norm $|a+ib| = \sqrt{a^2+b^2}$ for $a, b \in \mathbf{R}$.
- A *norm* on a $\mathbf{C}$-vector space V is a function $\|\cdot\|\colon V \to \mathbf{R}_+$ s.t.:
  - $\|v\| = 0$ iff $v = 0$;
  - $\|v+w\| \le \|v\| + \|w\|$ for $v, w \in V$;
  - $\|av\| = |a| \cdot \|v\|$ for $a \in \mathbf{C}$ and $v \in V$.
- If $\dim_{\mathbf{C}} V = 1$, any two norms on V are proportional, but there is no canonical norm on V in general.
- If $\pi\colon L \to X$ is a line bundle on a complex manifold X, then a *metric* on L is a function $\|\cdot\|\colon L \to \mathbf{R}_+$ whose restriction to $\pi^{-1}(x) \simeq \mathbf{C}$ is a norm for all $x \in X$.
- Use *additive* terminology and identify a metric $\|\cdot\|$ with

$$\phi := -\log\|\cdot\|\colon L^\times \to \mathbf{R},$$

  where $L^\times$ is L with the zero section removed.

Calculus on metrics on line bundles

- Use additive terminology on line bundles, too: $L_1 + L_2 := L_1 \otimes L_2$.
- ϕ_i metric on L_i, $a_i \in \mathbf{Z} \implies a_1\phi_1 + a_2\phi_2$ metric on $a_1L_1 + a_2L_2$.
- If $s \in \Gamma(U, L)$ is a local nowhere vanishing section, then $\phi := \log|s|$ is a metric on L over U for which $\phi \circ s \equiv 0$.
- Identify metrics on $\mathcal{O}_X$ with functions on X: evaluate at "1".
- Given a reference metric ϕ_0 on L, any other metric on L is of the form $\phi = \phi_0 + \varphi$, where φ is a function on X.
- Given a metric ϕ on L, set $dd^c\phi := dd^c(\phi \circ s)$ for any local nonvanishing section s of L. Then $dd^c\phi \in c_1(L)$.
- Say ϕ is *positive* if ϕ smooth and $dd^c\phi$ Kähler, i.e. $dd^c\phi > 0$.
- Any metric ϕ on K_X induces *volume form* $e^{2\phi}$ on X and conv'ly.
- In this way, $\operatorname{Ric}\omega = -dd^c\frac{1}{2}\log|\omega^n|$ for any Kähler form ω.
- X Fano, ϕ positive metric on $-K_X \implies dd^c\phi$ is a KE metric iff

$$(dd^c\phi)^n = c(\phi)e^{-2\phi}.$$

Functionals on the space of metrics

- Let (X, L) be a polarized smooth complex projective variety. Identify X and L with their analytifications.
- Redefine $\mathcal{H}$ as the space of *positive metrics* ϕ on L.
- Can define several *functionals* on $\mathcal{H}$.

  The Monge-Ampère energy and related functionals.
  The Mabuchi (or K-energy) functional.
  The Ding functional (in the Fano case $L = -K_X$).
- Set $V = (L^n)$. Then $\int_X (dd^c\phi)^n = V$ for all $\phi \in \mathcal{H}$.
- Given metric $\phi \in \mathcal{H}$, set

$$\mathrm{MA}(\phi) := V^{-1}(dd^c\phi)^n.$$

  This is a probabilility measure on X.

Monge-Ampère energy

- Fix reference metric $\phi_0 \in \mathcal{H}$. Any other metric is of the form $\phi = \phi_0 + \varphi$, where φ is a *function* on X.
- Define the *Monge-Ampère energy* of ϕ as

$$E(\phi) = \frac{1}{n+1} \sum_{j=0}^{n} V^{-1} \int_X \varphi (dd^c \phi)^j \wedge (dd^c \phi_0)^{n-j}$$

  where $V = (L^n)$.
- This is the *antiderivative of the Monge-Ampère* operator:

$$E'(\phi) = \mathrm{MA}(\phi)$$

  i.e. $\frac{d}{dt} E(\phi + tf)\big|_{t=0} = \int_X f \, \mathrm{MA}(\phi)$ for $f \in C^\infty(X)$.
- Also have $E(\phi_0) = 0$ and $E(\phi + c) = E(\phi) + c$ for $c \in \mathbf{R}$.
- Can get rid of reference metric by viewing E as a metric $\langle \phi^{n+1} \rangle$ on the line $\langle L^{n+1} \rangle$ given by the Deligne pairing.

The functionals I, J, and $I - J$

- Using the Monge-Ampère energy, can define several functionals that serve as "norms" or exhaustion functions on $\mathcal{H}$.
- The functionals I, J and $I - J$ are given by

$$I(\phi) = \int_X \varphi(\mathrm{MA}(\phi_0) - \mathrm{MA}(\phi))$$
$$J(\phi) = \int_X \varphi\, \mathrm{MA}(\phi_0) - E(\phi)$$
$$(I - J)(\phi) = E(\phi) - \int_X \varphi\, \mathrm{MA}(\phi).$$

- They are translation invariant: $I(\phi + c) = I(\phi)$ etc.
- We have $I(\phi) \geq 0$ with equality iff $\phi = \phi_0 + c$. Same for J, $I - J$.
- We have the inequality

$$n^{-1} J \leq I - J \leq nJ,$$

so the three functionals are equivalent.

The Ding functional

- Assume X is Fano and that $L = -K_X$.
- Any metric $\phi \in \mathcal{H}$ then induces a volume form $e^{-2\phi}$ on X.
- The Ding functional on $\mathcal{H}$ is defined by

$$D(\phi) = L(\phi) - E(\phi),$$

  where

$$L(\phi) = -\frac{1}{2} \log \int_X e^{-2\phi}.$$

- We have

$$D'(\phi) = e^{-2\phi} / \int_X e^{-2\phi} - \mathrm{MA}(\phi)$$

- Thus the critical points of Ding are Kähler-Einstein metrics!
- More precisely: $D'(\phi) = 0$ iff $\omega = dd^c\phi$ is a KE metric.

Entropy

- Define a reference prob. measure on X by $\mu_0 = e^{-2\phi_0} / \int_X e^{-2\phi_0}$, where $\phi_0 \in \mathcal{H}$ is the reference metric.
- Define the *entropy* of a probability measure μ (wrt μ_0) as

$$\mathrm{Ent}(\mu) := \int_X \log \frac{d\mu}{d\mu_0} \mu,$$

  if $\mu \ll \mu_0$, and $\mathrm{Ent}(\mu) = +\infty$ otherwise.
- We have $\mathrm{Ent}(\mu) \geq 0$ with equality iff $\mu = \mu_0$.
- The entropy functional is the Legendre dual of the functional L:

$$\mathrm{Ent}(\mu) = \sup\{L(\phi) - \int (\phi - \phi_0)\mu \mid \phi \text{ smooth metric on } L\}$$

$$L(\phi) = \inf\{\mathrm{Ent}(\mu) + \int (\phi - \phi_0)\mu \mid \mu \text{ prob measure on } X\}$$

The Mabuchi functional

- Define the *Mabuchi functional* on $\mathcal{H}$ by

$$M(\phi) = H(\phi) - (I - J)(\phi),$$

  where

$$H(\phi) = \frac{1}{2}\,\mathrm{Ent}(\mathrm{MA}(\phi))$$

- The critical points of the Mabuchi functional

$$M'(\phi) = 0,$$

  also give rise to KE metrics, just like for the Ding functional.
- Can define the Mabuchi functional for general polarizations using a different formula. In this case, the critical points define *constant scalar curvature Kähler metrics*.
- The formula above is due to Tian and Chen.

Coercivity and KE metrics

- For Fano manifolds w/o nontrivial vector fields, the existence of KE metrics can be detected by the Mabuchi and Ding functionals.
- Say M is *coercive* if $\exists \delta, C > 0$ such that

$$M \geq \delta J - C \quad \text{on } \mathcal{H}.$$

- **Thm** [Tian97,..., BBEGZ16, DR17] If $\mathrm{Aut}(X)$ finite, TFAE
  (i) X admits a KE metric;
  (ii) D is coercive;
  (iii) M is coercive.
- By [DR17], the theorem is also true if X has nontrivial vector fields if one replaces $J(\phi)$ in the coercivity condition by $J_G(\phi) := \inf\{J(g^*\phi) \mid \phi \in G\}$ where $G = \mathrm{Aut}^0(X)$.
- In these lectures, we shall focus on the case when $G = \{\mathrm{id}\}$.
- Will take Thm for granted, and relate coercivity to K-stability!

A version of the YTD conjecture

- Goal for rest of lectures is to explain the following result.
- **Thm** [Berman-Boucksom-J] For a Fano manifold X w/o nontrivial vector fields, TFAE
  (i) X admits a KE metric;
  (ii) The Ding functional D is coercive;
  (iii) The Mabuchi functional M is coercive;
  (iv) X is uniformly K-stable;
  (v) X is uniformly Ding-stable;
- Will take the equivalence of (i)–(iii) for granted.
- Need to explain (iv) and (v).
- Will outline proof of (iii) $\implies$ (iv) $\implies$ (v) $\implies$ (iii).

Part 2: K-stability from several points of view

In this part (X Fano):

- K-stability via test configurations for X.
- K-stability via anticanonical $\mathbf{Q}$-divisors on X.
- K-stability via (divisorial) valuations on X.
- K-stability via valuations on the cone $\mathbf{C}(X)$.
- Berkovich analytifications.
- Test configurations as non-Archimedean metrics.
- K-stability via functionals on non-Archimedean metrics.
- K-stability and Ding stability.

K-stability

- The notion of K-stability was introduced by Tian and Donaldson to understand obstructions for KE metrics.
- It is inspired by and related to stability in the sense of GIT.
- Its is algebraic in the sense that it works over any algebraically closed field of characteristic zero.
- Here, will explain K-stability from 5 points of view:
  (1) Test configurations for $(X, -K_X)$.
  (2) Singularities of special divisors in $|-mK_X|$, $m \gg 0$.
  (3) Divisorial valuations on X.
  (4) Valuations on the cone $\mathbf{C}(X)$ of X.
  (5) Non-Archimedean metrics on $-K_X$.
- Will be sloppy with Cartier divisors, $\mathbf{Q}$-Cartier divisors, line bundles,...

Test configurations

- Let (X, L) be a polarized variety. A *test-configuration* for (X, L) is essentially a 1-parameter degeneration of (X, L). Consists of:
  (1) a flat scheme $\mathcal{X} \to \mathbf{P}^1$ and a $\mathbf{Q}$-line bundle $\mathcal{L}$ on $\mathcal{X}$;
  (2) a $\mathbf{C}^*$-action on $(\mathcal{X}, \mathcal{L})$ lifting the action on $\mathbf{P}^1$;
  (3) a $\mathbf{C}^*$-equivariant isomorphism

$$(\mathcal{X} \setminus \mathcal{X}_0, \mathcal{L}|_{\mathcal{X}\setminus\mathcal{X}_0}) \overset{\sim}{\to} (X \times (\mathbf{P}^1 \setminus \{0\}), p_1^* L).$$

- **** PICTURE ****
- The test configuration is *normal* if $\mathcal{X}$ is normal. It is *ample/semiample/nef* if $\mathcal{L}$ is *relatively* ample/semiample/nef.